$$H = c\,\vec{p}\cdot\vec{\alpha} + \beta m \qquad [c = v/v_f \text{ for CMP setting}]$$

Clifford algebra

$\{\alpha_i, \alpha_j\} = 0 \quad i \neq j$

$\alpha_i^2 = \beta^2 = 1$

$\{\alpha_i, \beta\} = 0$

for low en. applications ($m >>> p$),

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{4\times 4}$$

$$= \sigma_x \otimes \sigma_i \qquad = \sigma_z \otimes \mathbb{1}$$

so dirac eqn becomes

$$\begin{pmatrix} m\mathbb{1} & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m\mathbb{1} \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = E \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

χ, ϕ = 2×1 vector

$\phi \rightarrow e^{\ominus}$ dof

$\chi \rightarrow$ positron dof

Gen, $\phi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\chi^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are spin $\uparrow$ species.

Similarly,

$\phi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ & $\chi^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are spin $\downarrow$ species.

$\chi \rightarrow \uparrow$ spin is same $\downarrow$ spin of $e^{\ominus}$ (or lack of down spin)

Reduction of dirac H for 2D into 2 eff-spin characterized H

Consider $H = v\vec{\alpha}\cdot\vec{p} + \beta m$

now $H\begin{pmatrix} f \\ 0 \\ 0 \\ g \end{pmatrix}$, where $f = \psi$ for $\uparrow e^{\ominus}$
$g = \psi$ for $\uparrow$ hole

$$= E\begin{pmatrix} f \\ 0 \\ 0 \\ g \end{pmatrix}$$

$$\Rightarrow v\vec{\sigma}\cdot\vec{p}\begin{pmatrix} 0 \\ g \end{pmatrix} + m \times \begin{pmatrix} f \\ 0 \end{pmatrix} = E\begin{pmatrix} f \\ 0 \end{pmatrix} \quad -(1)$$

$$v\vec{\sigma}\cdot\vec{p}\begin{pmatrix} f \\ 0 \end{pmatrix} - m \times \begin{pmatrix} 0 \\ g \end{pmatrix} = E\begin{pmatrix} 0 \\ g \end{pmatrix} \quad -(2)$$

(1)+(2), $$v\vec{\sigma}\cdot\vec{p}\begin{pmatrix} f \\ g \end{pmatrix} + m\sigma_z\begin{pmatrix} f \\ g \end{pmatrix} = E\begin{pmatrix} f \\ g \end{pmatrix}$$

$\therefore$ let $\begin{pmatrix} f \\ g \end{pmatrix} = \varphi$, then the effective $\uparrow$ spin hamiltonian is

$$h_\uparrow(x) = v\vec{\sigma}\cdot\vec{p} + m\sigma_z$$ [valid in 2D, 3D]

Similarly, for $\psi = (0\ f\ g\ 0)^T$

$$h_\downarrow(x) = (v\sigma_x p_x - v\sigma_y p_y + m\sigma_z)\begin{pmatrix} f \\ g \end{pmatrix} = E\begin{pmatrix} f \\ g \end{pmatrix}$$

opposite

Derivation

$$v\vec{\sigma}\cdot\vec{p}\begin{pmatrix} g \\ 0 \end{pmatrix} + m\begin{pmatrix} 0 \\ f \end{pmatrix} = E\begin{pmatrix} 0 \\ f \end{pmatrix} \quad -A$$

$$v\vec{\sigma}\cdot\vec{p}\begin{pmatrix}0\\f\end{pmatrix} - m\begin{pmatrix}g\\0\end{pmatrix} = E\begin{pmatrix}g\\0\end{pmatrix} \quad -B$$

A+B, $\quad v\vec{\sigma}\cdot\vec{p}\begin{pmatrix}g\\f\end{pmatrix} - m\sigma_z\begin{pmatrix}g\\f\end{pmatrix} = E\begin{pmatrix}g\\f\end{pmatrix}$

But we want spinor to have $\begin{pmatrix}f\\g\end{pmatrix}$ structure. so we'll multiply by σ_x to ge

$$(v\vec{\sigma}\cdot\vec{p})\cdot\sigma_x\cdot\begin{pmatrix}f\\g\end{pmatrix} - m\sigma_z\sigma_x\begin{pmatrix}f\\g\end{pmatrix} = E\sigma_x\begin{pmatrix}f\\g\end{pmatrix}$$

$$\Rightarrow \quad \sigma_x(v\vec{\sigma}\cdot\vec{p})\sigma_x\begin{pmatrix}f\\g\end{pmatrix} + m\sigma_z\begin{pmatrix}f\\g\end{pmatrix} = E\begin{pmatrix}f\\g\end{pmatrix}$$

$$\therefore h_\downarrow = \sigma_x(v\vec{\sigma}\cdot\vec{p})\sigma_x + m\sigma_z \quad [\text{in 2D, 3d}]$$

$$\boxed{\therefore h_{\uparrow\downarrow}(x) = vp_x\sigma_x \pm vp_y\sigma_y + m(x)\sigma_z} \quad (\text{in 2D})$$

$\therefore$ diff spins move in diff directions.

In 3D

$$h_\uparrow = v(\vec{p}\cdot\vec{\sigma}) + m\sigma_z = v(p_x\sigma_x + p_y\sigma_y + p_z\sigma_z) + m\sigma_z$$

$$h_\downarrow = \sigma_x(v\vec{p}\cdot\vec{\sigma})\sigma_x + m\sigma_z = vp_x\sigma_x - p_y\sigma_y - p_z\sigma_z + m\sigma_z$$

remember, spinors now have $e^\ominus$ & hole$^\ominus$ interpretation.

Helicity:

"projection of spin along direction of motion"

→ not an "intrinsic" property

$$h = \frac{\vec{S} \cdot \vec{P}}{|\vec{p}|} \qquad \text{in } 4\times4 \rightarrow \frac{1}{2}\begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix}$$

Chirality:-

conjured to tabulate all "covariant" forms of currents consistent with Lorentz transf.

$\bar{\psi}\,(4\times 4)\,\psi$

$\Downarrow$

matrix

where $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$

Define $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$

$$= \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$\gamma^0 = \beta \Rightarrow \gamma^{0^2} = 1$ & $\underline{\gamma^{0\dagger} = \gamma^0}$

$\gamma^i = \beta\alpha_i$

$\gamma^{i\dagger} = -\gamma^i$

"Chirality can be thought of as "inherent" trait of all the particles, while helicity depends on the momentum of the particle."

$\hookrightarrow$ you can change frames (Lorentz boost) & change chirality

$$\{\gamma_5,\gamma^\mu\} = 0 \quad \& \quad [\gamma_5, H] = 2m\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

$\hookrightarrow \therefore$ good quantum no. if $\underline{m=0}$

"Physical interpretatⁿ" $\rightarrow$ same as helicity operator when $m=0$

$P_L = \left(\dfrac{1-\gamma_5}{2}\right) \rightarrow$ left projector

$P_R = \left(\dfrac{1+\gamma_5}{2}\right) \rightarrow$ right "

Chirality is of interest for massive fermions too as "charged current weak interactⁿ" couples to left handed (-ve) chirality spinors only. (V-A theory)

Dirac eqn in massless limit

DE $\quad (\gamma^\mu \underbrace{P_\mu}_{\to\, i\partial_\mu} - m)\psi = 0$

set $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ or $\begin{pmatrix} u_1(p) \\ u_2(p) \end{pmatrix}$ $\qquad \gamma^i = \beta\alpha_i$

$$\Downarrow$$

$$\begin{pmatrix} (p^0 - m)\mathbb{1} & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(p^0+m)\mathbb{1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

which leads to

$$(p^0 - \vec{\sigma}\cdot\vec{p})(u_1+u_2) = m(u_1 - u_2)$$

$$(p^0 + \vec{\sigma}\cdot\vec{p})(u_1-u_2) = m(u_1 + u_2)$$

call $u_L = \frac{1}{2}(u_1 - u_2) \qquad u_R = \frac{1}{2}(u_1 + u_2)$

& $m \to 0$,

$$\boxed{\begin{aligned} p^0 u_R &= \vec{\sigma}\cdot\vec{p}\, u_R \\ p^0 u_L &= -\vec{\sigma}\cdot\vec{p}\, u_L \end{aligned}}$$

Weyl equations

Solution:-

$$p^{0\,2} u_{R,L} = (\vec{p}\cdot\vec{p})\, u_{R,L}$$

$\therefore$ for non-trivial soln $\boxed{p^{0\,2} = \vec{p}\cdot\vec{p}}$

rel. disp.

$\therefore \quad p^0 = \pm|\vec{p}|$

for $p^0 = +|\vec{p}|$ ($\therefore$ not antiparticles)

$$\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} u_r = +u_r \qquad \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} u_l = -u_l$$

helicity eigenvalue $= +\frac{1}{2}$ hel. eigenvalue $= -\frac{1}{2}$

"Right handed" "Left handed"

ex $\Rightarrow \bar{\nu}_e$ (right ex $\Rightarrow \nu_e$

Chirality :-

massless dirac $\not{p}\, u(p) = 0 = \not{p}\, v(p)$

+ve en. -ve. en.

$$= \sqrt{\frac{|\vec{p}|}{2}} \begin{pmatrix} \tilde{u} \\ \frac{1}{|p|}\vec{\sigma}\cdot\vec{p}\,\tilde{u} \end{pmatrix} \qquad \sqrt{\frac{|\vec{p}|}{2}} \begin{pmatrix} \frac{1}{|\vec{p}|}\vec{\sigma}\cdot\vec{p}\,\tilde{v} \\ \tilde{v} \end{pmatrix}$$

In $m=0$ limit, for +ve en. soln,

$$\gamma_5 = \vec{\Sigma}\cdot\hat{p}$$